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Gradient estimates for $u_t = \Delta F(u)$ on manifolds and some Liouville-type theorems

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ABSTRACT

In this paper, we first prove a localized Hamilton-type gradient estimate for the positive solutions of Porous Media type equations:

$$u_t = \Delta F(u),$$

with $F'(u) > 0$, on a complete Riemannian manifold with Ricci curvature bounded from below. In the second part, we study Fast Diffusion Equation (FDE) and Porous Media Equation (PME):

$$u_t = \Delta\left(u^p\right), \quad p > 0,$$

and obtain localized Hamilton-type gradient estimates for FDE and PME in a larger range of p than that for Aronson–Bénilan estimate, Harnack inequalities and Cauchy problems in the literature. Applying the localized gradient estimates for FDE and PME, we prove some Liouville-type theorems for positive global solutions of FDE and PME on noncompact complete manifolds with nonnegative Ricci curvature, generalizing Yau's celebrated Liouville theorem for positive harmonic functions.

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1. Introduction

The goal of the paper is to establish a localized Hamilton-type gradient estimate for the positive solutions of Porous Media type equations, which are degenerate parabolic equations in general,

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$$u_t = \Delta F(u) \tag{1}$$

on a complete Riemannian manifold $(M^n, \mathbf{g})$ with $Ric(M^n) \geqslant -k$ for some $k \geqslant 0$. Here $F \in C^2(0,\infty)$, $F' > 0$, and Δ is the Laplace–Beltrami operator of the metric $\mathbf{g}$. Eq. (1) is a nonlinear version of the classical heat equation (case $F(u) = u$). Porous Media Equation (PME for short) (case $F(u) = u^p$, $p > 1$) has arisen in different applications to model diffusive phenomena like groundwater infiltration (Boussinesq's model, $p = 2$), flow of gas in porous media (Leibenzon–Muskat model, $p \geqslant 2$), heat radiation in plasmas ($m > 4$), and others. The mathematical theory started in the 1950's and got momentum in recent decades as a nonlinear diffusion problem with interesting geometrical aspects (free boundaries) and peculiar functional analysis. We refer to the monographs [11,12] for an account of the rather complete theory concerning existence, uniqueness, regularity and asymptotic behavior of PME. Some of the existence, uniqueness and regularity properties hold true for the so-called Fast Diffusion Equation (FDE for short) (case $F(u) = u^p$, $p \in (0,1)$). FDE appears in plasma physics and in geometric flows such as the Ricci flow on surfaces and the Yamabe flow, see in [4,11,12].

It is well known that, in the study of geometric analysis and other elliptic or parabolic equations, the gradient estimate and the Harnack inequality play an important role. The Li–Yau estimate and the Harnack inequality in the fundamental paper [7], where Li and Yau studied the heat equation on general Riemannian manifolds, have a tremendous impact on the field of geometric analysis. Since 1970s, Aronson–Bénilan estimate and the Harnack-type inequalities have been widely studied for PME and FDE defined on the whole Euclidean space, cf. [1–3,6]. Recently, Lu, Ni, Vázquez and Villani [8] studied PME and FDE on manifolds and got some localized Aronson–Bénilan estimates. The first Harnack-type inequality dealing with the Porous Media type equation (1) was attributed to S.T. Yau [13].

Theorem A. *(See S.T. Yau [13].) Let M^n be a compact Riemannian manifold without boundary, Ricci$(M) \geqslant 0$. Suppose that $F \in C^2(0,\infty)$ with $F' > 0$, $c(t) \in C^1(0,\infty)$, and u is any positive solution of the degenerate parabolic equation*

$$u_t = \Delta F(u) \tag{2}$$

on M^n. Let $\alpha \neq 0$ be an arbitrary constant. Define a function G on $(0,\infty)$ by $G'(s) = F'(s)/s$, and we abbreviate $G = G(u)$, $F^\kappa = F^\kappa(u)$, $\kappa = 0,1,2$.

If the conditions below are satisfied:

(A) $|\nabla G|^2 - \alpha G_t - c(t) \leqslant 0$ *at* $t = 0$;
(B) (*nonlinear condition*) *the following quadratic inequality holds true for all* $x \geqslant 0$

$$\begin{aligned} 0 \geqslant{}& \frac{1-\alpha}{\alpha^2}\left(\alpha u F'' - \frac{2(1-\alpha)}{n}F'\right)x^2 + \left(\frac{4(1-\alpha)}{n\alpha^2} - \frac{uF''}{F'}\right)c(t)x \\ &- \left(\frac{2}{n} + \alpha\frac{uF''}{F'}\right)\frac{c^2(t)}{\alpha^2 F'} - c'(t), \end{aligned}$$

then we have for all $t > 0$ that

$$|\nabla G|^2 - \alpha G_t - c(t) \leqslant 0.$$

For the heat equation on compact manifolds without boundary, Hamilton [5] studied another type of gradient estimates as:

Theorem B. *(See Hamilton [5].) Let M^n be a compact manifold without boundary and with $Ric(M^n) \geqslant -k$ for some $k \geqslant 0$. Let u be a smooth positive solution of the heat equation with $u \leqslant M$ for all $(x,t) \in M^n \times (0,\infty)$. Then*

$$\frac{|\nabla u|^2}{u^2} \leqslant \left(\frac{1}{t} + 2k\right) \ln \frac{M}{u}.$$

The Hamilton-type gradient estimate takes up a significant position in the study of the heat equation. However, the classical Hamilton's estimate is a global result which requires the heat equation to be posed on compact manifold without boundary. Recently, a localized Hamilton type gradient estimate was proven by Souplet and Zhang [10], which can be viewed as a combination of Li–Yau estimate and Hamilton's gradient estimate.

Theorem C. *(See Souplet and Zhang [10].) Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that $F \in C^2(0,\infty)$ with $F' > 0$, and u is any positive solution of the degenerate parabolic equation (1) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Suppose also that $u \leqslant M$ in $Q_{R,T}$. Then there exists a dimensional constant C such that*

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla_x u(x,t)|}{u(x,t)} \leqslant C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\left(1 + \ln \frac{M}{u}\right).$$

Moreover, if M^n has nonnegative Ricci curvature and u is any positive solution of the heat equation on $M \times (0,\infty)$, then there exist dimensional constants C_1, C_2 such that

$$\frac{|\nabla_x u(x,t)|}{u(x,t)} \leqslant C_1 \frac{1}{\sqrt{t}}\left(C_2 + \ln \frac{u(x,2t)}{u(x,t)}\right)$$

for all $x \in M^n$ and $t > 0$.

It is natural to seek a localized Hamilton-type gradient estimate for Porous Media type equation (1) as Souplet and Zhang [10] did for the heat equation on a complete manifold. Recently, Ma, Zhao and Song [9] proved a localized Hamilton-type gradient estimate for Eq. (1) under some strong assumptions [9, Theorem 7], where the gradient estimate for FDE ($u_t = \Delta(u^p)$) holds only for dimension $n = 2, 3$ with $p \in (1 - 1/\sqrt{n}, 1)$ [9, Corollary 9]. Such restrictions on p and n are unnatural and inadequate for applications since the mathematical theory of PME and FDE based on a priori estimates such as Aronson–Bénilan estimate and others (cf. [1,3,6,8], etc.), applies to all positive smooth solutions of PME and FDE on the condition that $p > p_c := 1 - 2/n$ for any dimension n.

The first main result of this paper is the following localized Hamilton-type gradient estimate for the Porous Media type equation (1), which generalizes Theorem C of Souplet and Zhang [10] for the heat equation, under suitable conditions on F, as Theorem A on the Harnack-type inequality for the Porous Media type equation (1) by Yau [13]:

Theorem 1.1 *(Gradient estimate). Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that $F \in C^2(\mathbf{R}^+)$ with $F' > 0$, and u is any positive solution of Eq. (1) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Define a function G on $(0,\infty)$ by $G'(s) = F'(s)/s$. Denote $\mathcal{U} = \mathrm{range}_{(x,t)\in Q_{R,T}} u(x,t) \subset (0,\infty)$. Choose nonnegative constants K, α, δ and τ such that*

$$K \geqslant \sup_{s\in\mathcal{U}} F'(s), \qquad \alpha - \sup_{s\in\mathcal{U}} G(s) \geqslant \delta > 0; \qquad \tau \geqslant \sup_{s\in\mathcal{U}} \left|sF''(s)\right|/F'(s).$$

If there exists a nonnegative constant γ, such that the condition below is satisfied:

$$\text{(C)}: \quad 2 + \frac{sF''(s)}{F'(s)}\left(2 - (n-1)\frac{sF''(s)}{F'(s)} \times \frac{\alpha - G(s)}{F'(s)}\right) \geqslant \gamma > 0, \quad \forall s \in \mathcal{U},$$

then there exists a constant $C(n, \delta, K, \tau, \gamma)$ depending only on n, δ, K, τ and γ such that

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leqslant C(n,\delta,K,\tau,\gamma)\left(\frac{1}{R}+\frac{1}{\sqrt{T}}+\sqrt{k}\right). \tag{3}$$

When $n=1$, the Ricci curvature lower bound k vanishes.

In the second part of this paper, we study Fast Diffusion Equation (FDE for short) and Porous Media Equation (PME for short):

$$u_t = \Delta\big(u^p\big), \quad p>0, \tag{4}$$

on a complete Riemannian manifold $(M^n, \mathbf{g})$ with $Ric(M^n) \geqslant -k$ for some $k \geqslant 0$.

Firstly for FDE, i.e. Eq. (4) with $p<1$, we obtain the localized Hamilton-type gradient estimate, which generalizes the gradient estimate for FDE (4) in [9] (Corollary 9 in [9]), where the estimate holds only for dimension $n=2$ or 3 with $p \in (1-1/\sqrt{n}, 1)$.

Theorem 1.2. *Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that $u \leqslant M$ is a positive solution of FDE (4) with*

$$1-\frac{4}{n+3} < p < 1, \quad \textit{for}\ n \geqslant 1,$$

in $Q_{R,T} \equiv B(x_0,R)\times[t_0-T,t_0] \subset M^n \times (-\infty,\infty)$, then there exists a constant $C(n,p)$ depending only on n and p such that

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C(n,p)\left(\frac{1}{R}+\frac{M^{(1-p)/2}}{\sqrt{T}}+\sqrt{k}\right). \tag{5}$$

When $n=1$, the Ricci curvature lower bound k vanishes.

Remark 1.1. One should notice that the range of p here is $(1-\frac{4}{n+3}, 1)$, while previous results on Aronson–Bénilan estimate and the Harnack-type differential inequalities for FDE (cf. [1,3,6,8], etc.) require $p \in ((1-\frac{2}{n})_+, 1)$. We can see that for $n \geqslant 3$, the range of p for our gradient estimate is larger than that in previous results [1,3,6,9]. Our gradient estimate will be a useful tool to study related problem for FDE in this large range of p, in which one couldn't deal with in the literature even on $\mathbf{R}^n$.

On a complete noncompact manifold with nonnegative Ricci curvature, an immediate application of Theorem 1.2 is the following time-dependent Liouville theorem for positive global solutions of FDE, generalizing Yau's celebrated Liouville theorem for positive harmonic functions, which states that any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is a constant function.

Theorem 1.3 *(Liouville theorem). Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. Let u be a positive ancient solution, a solution defined in all space and negative time, of FDE (4) for $1-\frac{4}{n+3} < p < 1$. If there is a strictly increasing function $L(s) \in C(\mathbf{R})$ with $L(s) \to \infty$ as $s \to \infty$, such that*

$$u(x,t) = o\big(L\big(d(x)\big) + |t|^{1/(1-p)}\big)$$

near infinity, then u is a constant function on M^n.

Remark 1.2. One might see that the growth condition in the spatial direction in theorem is very weak, since we might choose

$$L(s) = \exp\bigl(\exp\bigl(\cdots\bigl(\exp(s)\bigr)\cdots\bigr)\bigr)$$

with l exp for any $l > 0$. Note that one might write any positive harmonic function $v(x)$ as a positive global solution of $\Delta(u^p) = 0$ with $u(x) = v(x)^{1/p}$, hence Yau's celebrated Liouville theorem for positive harmonic functions is a special case of Theorem 1.3 for time-dependent positive solutions of FDE, while one couldn't do this for the Heat Equation (see examples in [10]).

Secondly for PME, i.e. Eq. (4) with $p > 1$, we first obtain the localized Hamilton-type gradient estimate for dimension $n = 1$:

Theorem 1.4. *Suppose that $u \leqslant M$ is a positive solution of PME (4) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$ with $n = 1$. Let $G(u) = \frac{p}{p-1}u^{p-1}$, $\alpha = \frac{p}{p-1}M^{p-1}(1+\delta)$ with some constant $\delta > 0$. Then for any $p > 1$, there exists a constant $C(p)$ depending only on p such that*

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leqslant C(p)\left(\frac{1+\delta}{\delta R} + \frac{1}{\sqrt{M^{p-1}\delta T}}\right).$$

An immediate application of the above gradient estimate is the following time-dependent Liouville theorem for PME with $p > 1$ on $\mathbf{R}$.

Theorem 1.5 *(Liouville theorem). Let u be a positive ancient solution, a solution defined in all space and negative time, to PME ($p > 1$) on $\mathbf{R}$, such that*

$$u(x,t) = o\bigl(d(x)^{1/(p-1)} + |t|^{1/(p-1)}\bigr)$$

near infinity. Then u is a constant.

And for PME on a complete Riemannian manifold (M^n, g) with $n \geqslant 2$, we obtain the localized Hamilton-type gradient estimate:

Theorem 1.6. *Let M be a complete Riemannian manifold with dimension $n \geqslant 2$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that u is a positive solution of PME (4) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$, with* $\mathrm{range}(u) = [m, M]$. *Let $G(u) = \frac{p}{p-1}u^{p-1}$, $\alpha = \frac{p}{p-1}M^{p-1}(1+\delta)$ with some small constant $0 < \delta \leqslant \frac{4}{n-1}$. If the following* pinch condition *on m, M holds*

$$1 \leqslant \left(\frac{M}{m}\right)^{p-1} < \frac{1}{1+\delta}\left(\frac{4p}{(n-1)(p-1)} + 1\right),$$

then there exists a constant $C(n, p)$ depending only on n and p, and

$$\gamma = 2p - \frac{(n-1)(p-1)}{2}\frac{M^{p-1}(1+\delta) - m^{p-1}}{m^{p-1}} > 0,$$

such that

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leqslant C(n,p)\left(\frac{\delta+1}{\gamma\delta R} + \frac{1}{\sqrt{\gamma\delta M^{p-1}T}} + \sqrt{\frac{k}{\delta}}\right).$$

The rest of this paper is organized as follows. In Section 2, we first derive a differential inequality for $w = |\nabla G|^2/(\alpha - G)^2$, where we use the technical linear algebra lemma to reduce the use of Cauchy–Schwarz inequality when we derive the differential inequality, while the free use of Cauchy–Schwarz inequality in their proof resulted that Theorem 7 in [9] required much stronger conditions. Finally we apply the maximum principle on the differential inequality to prove Theorem 1.1 following the argument in [10] by using the well-known cut-off function by Li and Yau in [7]. In Section 3, we apply Theorem 1.1 to study the localized Hamilton-type gradient estimates and Liouville-type theorems for FDE and PME and prove Theorems 1.2–1.6. In Appendix A, we prove the technical linear algebra lemma.

Here and later $Ric(M^n)$ is the Ricci curvature and a manifold is complete if every geodesic extends to infinity. We use the Einstein summation convention for indices i, j, k, etc. In particular, we use the short-hand notation $g_{ij}^2 = g_{ij}g_{ij} = \sum_{i,j=1}^n g_{ij}^2$. Hereafter we always use C to denote different constants depending on n only.

2. Hamilton-type gradient estimates

In this section, we prove Theorem 1.1 and show that the trick introduced in the fundamental work of Li and Yau [7] can be adapted to study the Hamilton-type estimate. Our argument can be considered as an improvement of the argument in [9], which can be regarded as a combination of the analysis in [7] and [10]. We define a quantity $w(x,t)$ and start with deriving a differential inequality on $w(x,t)$, then use the well-known cut-off function of Li and Yau [7], to derive the desired bounds.

Let $\phi = \ln u$. Since u is a solution to the equation $u_t = \Delta(F(u))$, simple calculation shows

$$\phi_t = \Delta\big(G(u)\big) + \nabla G(u)\cdot\nabla\phi. \tag{6}$$

Denote $g(\phi) = G(e^\phi)$, and multiply (6) by $g'(\phi)$, with some elementary computations, we get

$$\begin{gathered} g_t = g'\Delta g + |\nabla g|^2, \qquad \nabla g = g'\nabla\phi, \\ g'(\phi) = G'\big(e^\phi\big)e^\phi = F'(u), \\ g''(\phi) = F''\big(e^\phi\big)e^\phi = F''(u)u. \end{gathered}$$

Set

$$w = w(x,t) = \big|\nabla\ln(\alpha - g)\big|^2 = \frac{|\nabla g|^2}{(\alpha - g)^2}, \tag{7}$$

and we first derive the differential inequality for w, to which we apply the maximum principle.

Lemma 2.1. *w satisfies the following differential inequality*:

$$g'\Delta w - w_t \geqslant Lw^2 - 2g'kw - L_1\nabla g\cdot\nabla w,$$

where L and L_1 are some functions given in (15) *and* (16).

Proof. After some elementary computations in local orthonormal system as in [9], we get that

$$\begin{aligned} w_t &= 2\frac{\nabla g\cdot\nabla g_t}{(\alpha - g)^2} + 2\frac{|\nabla g|^2 g_t}{(\alpha - g)^3} \\ &= 2\frac{\nabla g\cdot\nabla(g'\Delta g + |\nabla g|^2)}{(\alpha - g)^2} + 2\frac{|\nabla g|^2(g'\Delta g + |\nabla g|^2)}{(\alpha - g)^3} \end{aligned}$$

$$=\frac{2g'\nabla g\cdot\nabla\Delta g}{(\alpha-g)^2}+\frac{2\Delta g\nabla g\cdot\nabla g'}{(\alpha-g)^2}+\frac{2\nabla g\cdot\nabla|\nabla g|^2}{(\alpha-g)^2}+\frac{2g'|\nabla g|^2\Delta g}{(\alpha-g)^3}+\frac{2|\nabla g|^4}{(\alpha-g)^3}$$

$$=2g'\frac{g_jg_{iij}}{(\alpha-g)^2}+2\frac{g''}{g'}\frac{|\nabla g|^2\Delta g}{(\alpha-g)^2}+4\frac{g_ig_{ij}g_j}{(\alpha-g)^2}+2g'\frac{|\nabla g|^2\Delta g}{(\alpha-g)^3}+2\frac{|\nabla g|^4}{(\alpha-g)^3}, \tag{8}$$

$$w_j=\left(\frac{g_i^2}{(\alpha-g)^2}\right)_j=2\frac{g_ig_{ij}}{(\alpha-g)^2}+2\frac{g_i^2g_j}{(\alpha-g)^3}, \tag{9}$$

$$\begin{aligned}\Delta w=w_{jj}&=2\left(\frac{g_ig_{ij}}{(\alpha-g)^2}\right)_j+2\left(\frac{g_i^2g_j}{(\alpha-g)^3}\right)_j\\&=2\frac{g_{ij}^2}{(\alpha-g)^2}+2\frac{g_ig_{ijj}}{(\alpha-g)^2}+8\frac{g_ig_{ij}g_j}{(\alpha-g)^3}+2\frac{|\nabla g|^2\Delta g}{(\alpha-g)^3}+6\frac{|\nabla g|^4}{(\alpha-g)^4}.\end{aligned} \tag{10}$$

By (8) and (10), we obtain that

$$\begin{aligned}g'\Delta w-w_t={}&2g'\frac{g_{ij}^2}{(\alpha-g)^2}+2g'\frac{g_ig_{ijj}-g_jg_{iij}}{(\alpha-g)^2}+8g'\frac{g_ig_{ij}g_j}{(\alpha-g)^3}\\&-4\frac{g_ig_{ij}g_j}{(\alpha-g)^2}+6g'\frac{|\nabla g|^4}{(\alpha-g)^4}-2\frac{g''}{g'}\frac{|\nabla g|^2\Delta g}{(\alpha-g)^2}-2\frac{|\nabla g|^4}{(\alpha-g)^3}.\end{aligned}$$

Bochner's identity implies that

$$g_ig_{ijj}-g_jg_{iij}=g_j(g_{jii}-g_{iij})=R_{ij}g_ig_j=Ric(\nabla g,\nabla g)$$

where R_{ij} is the Ricci curvature tensor. Therefore we have

$$\begin{aligned}g'\Delta w-w_t={}&2g'\frac{g_{ij}^2}{(\alpha-g)^2}+2g'\frac{Ric(\nabla g,\nabla g)}{(\alpha-g)^2}+8g'\frac{g_ig_{ij}g_j}{(\alpha-g)^3}-4\frac{g_ig_{ij}g_j}{(\alpha-g)^2}\\&+6g'\frac{|\nabla g|^4}{(\alpha-g)^4}-2\frac{g''}{g'}\frac{|\nabla g|^2\Delta g}{(\alpha-g)^2}-2\frac{|\nabla g|^4}{(\alpha-g)^3}.\end{aligned} \tag{11}$$

Recalling (9), we have

$$\nabla g\cdot\nabla w=2\frac{g_ig_{ij}g_j}{(\alpha-g)^2}+2\frac{|\nabla g|^4}{(\alpha-g)^3}. \tag{12}$$

Adding $(2-\frac{2g'}{\alpha-g}-\eta\frac{g''}{g'})\times(12)$ with (11), where η is a parameter function to be determined later, we conclude that

$$\begin{aligned}g'\Delta w-w_t={}&2g'\frac{g_{ij}^2}{(\alpha-g)^2}+2\left(\frac{2g'}{\alpha-g}-\eta\frac{g''}{g'}\right)\frac{g_ig_{ij}g_j}{(\alpha-g)^2}-2\frac{g''}{g'}\frac{|\nabla g|^2\Delta g}{(\alpha-g)^2}\\&+\left(2g'+\left[2-2\eta\frac{g''}{g'}\right](\alpha-g)\right)\frac{|\nabla g|^4}{(\alpha-g)^4}\\&+2g'\frac{Ric(\nabla g,\nabla g)}{(\alpha-g)^2}-\left(2-\frac{2g'}{\alpha-g}-\eta\frac{g''}{g'}\right)\nabla g\cdot\nabla w.\end{aligned} \tag{13}$$

Denote $f = 2g'/(\alpha - g)$, $b = g''/g'$, $A = (g_{ij})$ and $\nu = \nabla g/|\nabla g|$, then we have $g_{ij}^2 = |A|^2$, $g_i g_{ij} g_j = A(\nu, \nu)|\nabla g|^2$ and $\Delta g = \operatorname{tr} A$. From definition of w, the right side of (13) can be written as

$$\begin{aligned}
&2g'\frac{|A|^2}{(\alpha - g)^2} + 2(\alpha - g)\left[(f - \eta b)\frac{A(\nu,\nu)}{|A|} - b\frac{\operatorname{tr} A}{|A|}\right]\frac{|A|}{\alpha - g}w \\
&\quad + (\alpha - g)(f + 2 - 2\eta b)w^2 + 2g' Ric(\nu,\nu)w - (2 - f - \eta b)\nabla g \cdot \nabla w \\
&\quad = 2g'\left[\frac{|A|}{(\alpha - g)} + \frac{1}{f}\left((f - \eta b)\frac{A(\nu,\nu)}{|A|} - b\frac{\operatorname{tr} A}{|A|}\right)w\right]^2 \\
&\quad + (\alpha - g)\left(f + 2 - 2\eta b - \frac{1}{f}\left[(f - \eta b)\frac{A(\nu,\nu)}{|A|} - b\frac{\operatorname{tr} A}{|A|}\right]^2\right)w^2 \\
&\quad + 2g' Ric(\nu,\nu)w - (2 - f - 2\eta b)\nabla g \cdot \nabla w \\
&\quad \geqslant \frac{\alpha - g}{f}\left(f^2 + [2 - 2\eta b]f - \left[(f - \eta b)\frac{A(\nu,\nu)}{|A|} - b\frac{\operatorname{tr} A}{|A|}\right]^2\right)w^2 \\
&\quad + 2g' Ric(\nu,\nu)w - (2 - f - \eta b)\nabla g \cdot \nabla w.
\end{aligned} \tag{14}$$

Applying Lemma A.1 from Appendix A to (14), we have

$$\begin{aligned}
g'\Delta w - w_t &\geqslant \frac{\alpha - g}{f}\big(f^2 + (2 - 2\eta b)f - (f - \eta b - b)^2 - (n-1)b^2\big)w^2 \\
&\quad + 2g' Ric(\nu,\nu)w - (2 - f - \eta b)\nabla g \cdot \nabla w \\
&= \frac{\alpha - g}{f}\big(2(1 + b)f - \big(n - 1 + [\eta + 1]^2\big)b^2\big)w^2 \\
&\quad + 2g' Ric(\nu,\nu)w - (2 - f - \eta b)\nabla g \cdot \nabla w.
\end{aligned}$$

Next we estimate the coefficient L of w^2. Pick $\eta = -1$. We can bound L as the following:

$$L = (\alpha - g)\left(2(1 + b) - (n-1)\frac{b^2}{f}\right) \geqslant (\alpha - g)\gamma > 0, \tag{15}$$

since $2(1 + b) - (n-1)\frac{b^2}{f} \geqslant \gamma > 0$ from our condition (C) in Theorem 1.1. We estimate the coefficient L_1 of $\nabla g \cdot \nabla w$ as

$$|L_1| = |2 - f + b| \leqslant 2 + \tau + f. \tag{16}$$

Hence, from $Ric(M^n) \geqslant -k$, we have

$$g'\Delta w - w_t \geqslant Lw^2 - 2g'kw - L_1\nabla g \cdot \nabla w. \qquad \square \tag{17}$$

Now we can apply maximum principle to the differential inequality (17) to prove our gradient estimate (3). We will follow [9] and [10] to use the well-known cut-off function by Li and Yau [7] to show Theorem 1.1. We caution the reader that the calculation is not the same as that in [7] due to the difference of the first-order term.

Proof of Theorem 1.1. Let $\Psi = \Psi(x,t)$ be a smooth cut-off function supported in $Q_{R,T}$, satisfying the following properties:

(1) $\Psi = \Psi(d(x, x_0), t)$, $\Psi = 1$ in $Q_{R/2,T/2}$, $0 \leqslant \Psi \leqslant 1$;
(2) Ψ is decreasing as a radial function in the spatial variables;
(3) $\frac{\partial_r \Psi}{\Psi^a} \leqslant \frac{C_a}{R}$, $\frac{\partial_r^2 \Psi}{\Psi^a} \leqslant \frac{C_a}{R^2}$, when $0 < a < 1$;
(4) $\frac{|\partial_t \Psi|^2}{\Psi} \leqslant \frac{C}{T^2}$.

Then, from (17) and a straightforward calculation, one has

$$\begin{aligned} &g' \Delta(\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla(\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla(\Psi w) \\ &\quad \geqslant L\Psi w^2 - 2kg'\Psi w + g'\Delta\Psi w - \Psi_t w + L_1(\nabla g \cdot \nabla \Psi) w - 2g' \frac{|\nabla \Psi|^2}{\Psi} w. \end{aligned} \tag{18}$$

We obtain the upper bounds for each term of the right-hand side of (18) as did by Souplet and Zhang [10]:

$$\left|2kg'\Psi w\right| \leqslant \frac{1}{6} L\Psi w^2 + \frac{6k^2 g'^2 \Psi}{L} \leqslant \frac{1}{6} L\Psi w^2 + Ck^2 \frac{g'^2}{L}, \tag{19}$$

$$|\Psi_t w| \leqslant \frac{1}{6} L\Psi w^2 + \frac{3|\Psi_t|^2}{2L\Psi} \leqslant \frac{1}{6} L\Psi w^2 + \frac{C}{T^2}\frac{1}{L}, \tag{20}$$

$$\left|2g' \frac{|\nabla \Psi|^2}{\Psi} w\right| \leqslant \frac{1}{6} L\Psi w^2 + \frac{6g'^2}{L}\frac{|\nabla\Psi|^4}{\Psi^3} \leqslant \frac{1}{6} L\Psi w^2 + \frac{C}{R^4}\frac{g'^2}{L}, \tag{21}$$

$$\begin{aligned} \left|L_1(\nabla g \cdot \nabla \Psi) w\right| &= \left|L_1(\nabla g \cdot \nabla \Psi) \frac{\alpha - g}{|\nabla g|} w^{3/2}\right| \\ &\leqslant \frac{1}{6} L\Psi w^2 + \frac{C L_1^4 (\alpha - g)^4}{L^3} \frac{|\nabla \Psi|^4}{\Psi^3} \\ &\leqslant \frac{1}{6} L\Psi w^2 + \frac{C}{R^4} \frac{L_1^4(\alpha - g)^4}{L^3}. \end{aligned} \tag{22}$$

Here we use Young's inequality,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall p, q > 0 \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

Furthermore, by the properties of Ψ and the assumption on the Ricci curvature, one has

$$\begin{aligned} \left|-g'\Delta\Psi w\right| &\leqslant \left|g'\left[\partial_r^2 \Psi + (n-1)\frac{\partial_r \Psi}{r} + \partial_r \Psi \partial_r \ln(\sqrt{\mathbf{g}})\right] w\right| \\ &\leqslant g'\left[\left|\partial_r^2 \Psi\right| + 2(n-1)\frac{|\partial_r \Psi|}{R} + k|\partial_r \Psi|\right] w \\ &\leqslant \frac{1}{6} L\Psi w^2 + \frac{9g'^2}{L}\left[\frac{|\partial_r^2 \Psi|^2}{\Psi} + 4(n-1)^2 \frac{|\partial_r \Psi|^2}{R^2 \Psi} + k\frac{|\partial_r \Psi|^2}{\Psi}\right] \\ &\leqslant \frac{1}{6} L\Psi w^2 + C\left[\frac{1}{R^4} + \frac{1}{R^4} + \frac{k}{R^2}\right]\frac{g'^2}{L}. \end{aligned} \tag{23}$$

Inserting (19)–(23) into the right-hand side of (18), we deduce that

$$
\begin{aligned}
& g'\Delta(\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla(\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla(\Psi w) \\
& \quad \geqslant \frac{L}{6}\Psi w^2 - C\left[\frac{L_1^4(\alpha - g)^4}{L^3}\frac{1}{R^4} + \frac{g'^2}{L}\frac{1}{R^4} + \frac{1}{L}\frac{1}{T^2} + \frac{g'^2}{L}k^2\right] \\
& \quad \geqslant \frac{L}{6}\left[\Psi w^2 - C\left[\frac{L_1^4(\alpha - g)^4}{L^4}\frac{1}{R^4} + \frac{g'^2}{L^2}\frac{1}{R^4} + \frac{1}{L^2}\frac{1}{T^2} + \frac{g'^2}{L^2}k^2\right]\right].
\end{aligned} \tag{24}
$$

Recalling that $L \geqslant 2\gamma(\alpha - g) > 0$, $L_1 \leqslant 2 + \tau + f$ and $f = g'/(\alpha - g)$, therefore,

$$
\begin{aligned}
& g'\Delta(\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla(\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla(\Psi w) \\
& \quad \geqslant \frac{L}{6}\left[\Psi w^2 - C(\gamma)\left(\frac{(1+f+\tau)^4}{R^4} + \frac{1}{(\alpha - g)^2}\left[\frac{g'^2}{R^4} + \frac{1}{T^2} + g'^2 k^2\right]\right)\right] \\
& \quad \geqslant \frac{L}{6}\left[\Psi w^2 - C(\gamma)\left(\frac{(1+f+\tau)^4}{R^4} + \frac{1}{(\alpha - g)^2}\frac{1}{T^2} + f^2 k^2\right)\right].
\end{aligned} \tag{25}
$$

Suppose that the maximum of (Ψw) is reached at $(x_1, t_1) \in Q_{R,T}$. By [7], we can assume, without loss of generality, that x_1 is not in the cut-locus of M. Then at this point, one has $\Delta(\Psi w) \leqslant 0$, $(\Psi w)_t \geqslant 0$ and $\nabla(\Psi w) = 0$. Recalling that $\alpha - g \geqslant \delta$, $0 < g' \leqslant K$ and $f \leqslant K/\delta$, we have

$$
(\Psi w^2)(x_1, t_1) \leqslant C(\gamma)\left(\frac{(\delta + K + \tau\delta)^4}{\delta^4 R^4} + \frac{1}{\delta^2}\frac{1}{T^2} + \left(\frac{K}{\delta}\right)^2 k^2\right).
$$

By assumption, the maximum of (Ψw) is reached at $(x_1, t_1) \in Q_{R,T}$, which implies that for any $(x, t) \in Q_{R,T}$

$$
\begin{aligned}
(\Psi w)^2(x, t) & \leqslant (\Psi w)^2(x_1, t_1) \leqslant (\Psi w^2)(x_1, t_1) \\
& \leqslant C(\gamma)\left(\frac{(\delta + K + \tau\delta)^4}{\delta^4 R^4} + \frac{1}{\delta^2 T^2} + \left(\frac{K}{\delta}\right)^2 k^2\right).
\end{aligned}
$$

Noticing that $\Psi(x, t) = 1$ in $Q_{R/2,T/2}$ and $w = |\nabla g|^2/(\alpha - g)^2$, we finally have proven

$$
\frac{|\nabla g|^2}{(\alpha - g)^2} \leqslant C(\gamma)\left(\frac{(\delta + K + \tau\delta)^2}{\delta^2 R^2} + \frac{1}{\delta T} + \frac{K}{\delta}k\right)
$$

which is exactly what the conclusion of Theorem 1.1 is. □

3. Gradient estimates and Liouville theorems for FDE and PME

In this section, we study the heat equation, FDE and PME on a complete Riemannian manifold, and derive the localized Hamilton-type gradient estimates by applying our Theorem 1.1, then prove some time-dependent Liouville theorems for FDE and PME on noncompact complete manifolds with nonnegative Ricci curvature. As a corollary, we obtain Yau's celebrated Liouville theorem for positive harmonic functions: *any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is a constant function.*

3.1. Heat equations

Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that $u \leqslant M$ is a positive solution of the heat equation

$$u_t = \Delta u$$

in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. We may look as the case of $F(u) = u$ in Theorem 1.1. Choose $G(s) = \ln s$ and let $\alpha = 1 + \ln M$, $K = 1$, $\tau = 1$ and $\gamma = 2$ in Theorem 1.1, we obtain the main result of Souplet and Zhang [10], Theorem C, as a direct corollary of our Theorem 1.1.

Using the localized Hamilton-type gradient estimates for positive solutions of the heat equation, Souplet and Zhang [10] proved the following time-dependent Liouville theorem:

Corollary 3.1. *(See Theorem 1.2 in [10].) Let M be a complete, noncompact manifold with nonnegative Ricci curvature. Then the following conclusions hold.*

(a) *Let u be a positive ancient solution to the heat equation (that is, a solution defined in all space and negative time) such that $u(x,t) = e^{o(d(x)+\sqrt{|t|})}$ near infinity. Then u is a constant.*
(b) *Let u be an ancient solution to the heat equation such that $u(x,t) = o([d(x) + \sqrt{|t|}])$ near infinity. Then u is a constant.*

As discussed in [10], one could not expect that Yau's Liouville theorem would still hold for positive ancient or eternal solutions to the heat equation. Both growth conditions of the above theorem in the spatial direction are sharp, by some simple examples, i.e., for (a), let $u(x,t) = e^{x+t}$ on $x \in \mathbf{R}$, and for (b), let $u(x,t) = x$ on $x \in \mathbf{R}$. Hence one couldn't obtain Yau's celebrated Liouville theorem for positive harmonic functions directly from the above time-dependent Liouville theorem for heat equation.

3.2. Fast diffusion equation

Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Here we consider the Fast Diffusion Equation (FDE for short)

$$u_t = \Delta\big(u^p\big), \quad 0 < p < 1,$$

on $M^n \times (-\infty, \infty)$. We have the following localized Hamilton-type gradient estimates for the positive solution on $M^n \times (-\infty, \infty)$:

Theorem 3.1. *Suppose that $u \leqslant M$ is a positive solution of the FDE*

$$u_t = \Delta\big(u^p\big)$$

in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$, where

$$1 - \frac{4}{n+3} < p < 1, \quad \textit{for } n \geqslant 1. \tag{26}$$

Then there exists a constant C depending only on n and p such that

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C\left(\frac{1}{R} + \frac{M^{(1-p)/2}}{\sqrt{T}} + \sqrt{k}\right).$$

When $n = 1$, the Ricci curvature lower bound k vanishes.

Proof. For $0 < p < 1$, we have $F(s) = s^p$, and choose $G(s) = \frac{p}{p-1}s^{p-1}$, and let $\alpha = 0$ and $\tau \geqslant 1 - p$ in Theorem 1.1. We have $F'(s) = ps^{p-1}$, $F''(s) = p(p-1)s^{p-2}$, then condition (C) in Theorem 1.1 becomes

$$4p - (n-1)(1-p) > 0,$$

which is equivalent to

$$1 - \frac{4}{n+3} < p < 1.$$

Let $\gamma = ((n+3)p - (n-1))/2$ for given p in admission range (26). Following the proof of Theorem 1.1, we have

$$\begin{cases} g = \dfrac{p}{p-1}u^{p-1}, \quad g' = pu^{p-1}, \quad g'' = p(p-1)u^{p-1}, \\ f = \dfrac{2g'}{-g} = 2(1-p), \quad b = \dfrac{g''}{g'} = p-1, \\ L = \gamma\dfrac{p}{1-p}u^{p-1}, \quad L_1 = 3-p. \end{cases}$$

We follow the proof of Theorem 1.1 until (24) with the modification that the constant $C(n, p)$ here depends only on n and p. Inserting the above quantities into (24), we have

$$\begin{aligned} &2g'\Delta(\Psi w) - (\Psi w)_t + L_1\nabla g \cdot \nabla(\Psi w) - 2g'\frac{\nabla\Psi}{\Psi} \cdot \nabla(\Psi w) \\ &\quad \geqslant \frac{L}{6}\left[\Psi w^2 - C(n, p)\left(\frac{1}{R^4} + \frac{1}{u^{2(p-1)}}\frac{1}{T^2} + k^2\right)\right]. \end{aligned}$$

By the same maximum argument in the proof of Theorem 1.1, we have

$$\Psi w^2 \leqslant C(n, p)\left(\frac{1}{R^4} + \frac{M^{2(1-p)}}{T^2} + k^2\right),$$

which implies

$$\frac{|\nabla g|^2}{(-g)^2} \leqslant C(n, p)\left(\frac{1}{R^2} + \frac{M^{1-p}}{T} + k\right).$$

Then the conclusion of Theorem 3.1 follows easily from the fact that

$$\frac{|\nabla g|}{-g} = \frac{|\nabla G(u)|}{-G(u)} = (1-p)\frac{|\nabla u|}{u}. \qquad \square$$

An immediate application of the above gradient estimates is the following time-dependent Liouville theorem for FDE on a complete noncompact manifold with nonnegative Ricci curvature:

Theorem 3.2 *(Liouville theorem). Let M^n be a complete, noncompact manifold with nonnegative Ricci curvature. Let u be a positive ancient solution, a solution defined in all space and negative time, of the Fast Diffusion Equation for $1 - \frac{4}{n+3} < p < 1$, and $L(s) \in C(\mathbf{R})$ be any strictly increasing function with $L(s) \to \infty$ as $s \to \infty$, such that*

$$u(x,t) = o\big(L\big(d(x)\big) + |t|^{1/(1-p)}\big)$$

near infinity. Then u is a constant.

Proof. Since $L(s)$ is a strictly increasing function with $L(s) \to \infty$ as $s \to \infty$, there is an inverse function $H(s)$ of $L(s)$ which is also a strictly increasing function with $H(s) \to \infty$ as $s \to \infty$. Fixing (x_0, t_0) in space–time and using Theorem 3.1 for u on the cube $Q(\frac{1}{2}H(R^{2/(1-p)}), R^2) = B(x_0, \frac{1}{2}H(R^{2/(1-p)})) \times [t_0 - R^2, t_0]$, and the M in Theorem 3.1 is the maximum value on the double cube $Q(H(R^{2/(1-p)}), 2R^2)$, by our assumption on the growth condition of the function u at infinity,

$$M_{H(R^{2/(1-p)}),2R^2} = o\big(L\big(H\big(R^{2/(1-p)}\big)\big) + R^{2/(1-p)}\big) = o\big(R^{2/(1-p)}\big). \tag{27}$$

Hence by Theorem 3.1, we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)} \leqslant C\left(\frac{1}{\frac{1}{2}H(R^{2/(1-p)})} + \frac{1}{R}o(R)\right). \tag{28}$$

Letting $R \to \infty$, it follows that $|\nabla u(x_0,t_0)| = 0$. Since $(x_0,t_0) \in M^n \times \mathbf{R}$ is arbitrary, one sees that u must be a constant function. □

As a corollary of the above time-dependent Liouville theorem for FDE, we obtain Yau's celebrated Liouville theorem for positive harmonic functions on a complete, noncompact manifold with nonnegative Ricci curvature:

Corollary 3.2 *(Yau's Liouville theorem for positive harmonic functions). Any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is a constant function.*

Proof. The proof follows immediately from the above time-dependent Liouville theorem. Let v be a positive harmonic function. Choose a p with $1 - \frac{4}{n+3} < p < 1$, then $u(x) = v(x)^{1/p}$ is a positive solution of $\Delta(u^p) = 0$, which can be considered as a time-independent positive solution of the corresponding Fast Diffusion Equation. Define

$$L(s) = s \max_{d(x) \leqslant s} u(x) + s.$$

It is easy to see that $L(s)$ is a strictly increasing function with $L(s) \to \infty$ as $s \to \infty$, and $u(x,t) = o(L(d(x)))$ near infinity. Following from the above theorem, u must be a constant function, which implies v must be a constant function. □

3.3. Porous Media Equations

Let M^n be a complete Riemannian manifold with dimension $n \geqslant 1$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Here we consider Porous Media Equation (PME for short)

$$u_t = \Delta\big(u^p\big), \quad p > 1,$$

on $M^n \times (-\infty, \infty)$.

For dimension $n = 1$, we have the following localized Hamilton-type gradient estimates for the positive solution on $M^n \times (-\infty, \infty)$:

Theorem 3.3. *Suppose that $u \leqslant M$ is a positive solution of PME (4) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$ with $n = 1$. Let $G(u) = \frac{p}{p-1} u^{p-1}$, $\alpha = \frac{p}{p-1} M^{p-1}(1+\delta)$ with some constant $\delta > 0$. Then for any $p > 1$, there exists a constant $C(p)$ depending only on p such that*

$$\sup_{(x,t)\in Q_{R/2,T/2}} \frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leqslant C(p)\left(\frac{1+\delta}{\delta R} + \frac{1}{\sqrt{M^{p-1}\delta T}}\right).$$

Proof. We have $F(s) = s^p$ and choose $G(s) = \frac{p}{p-1} s^{p-1}$ in Theorem 1.1. We have $F'(s) = ps^{p-1}$, $F''(s) = p(p-1)s^{p-2}$, and $K = pM^{p-1}$, $\tau \geqslant p - 1$, and α as defined above, then condition (C) in Theorem 1.1 becomes

$$2p \geqslant \gamma > 0.$$

Condition (C) is satisfied if we choose $\gamma = 2p$. Similar to the proof of Theorem 3.1, we have

$$\begin{cases} g = \dfrac{p}{p-1} u^{p-1}, \quad g' = pu^{p-1}, \quad g'' = p(p-1)u^{p-1}, \\ f = \dfrac{2g'}{\alpha - g} = 2(p-1)\dfrac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leqslant \dfrac{2(p-1)}{\delta}, \\ b = \dfrac{g''}{g'} = p - 1, \\ L = \gamma(\alpha - g) = \dfrac{2p^2}{1-p}\left[M^{p-1}(1+\delta) - u^{p-1}\right] \geqslant \dfrac{2p^2}{1-p} M^{p-1}\delta, \\ L_1 = 1 + p + f \leqslant 1 + p + 2(p-1)\dfrac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leqslant 1 + p + \dfrac{2(p-1)}{\delta}. \end{cases}$$

By the same argument in the proof of Theorem 3.1, we have

$$\Psi w^2 \leqslant C(p)\left(\frac{L_1^4}{R^4} + \frac{1}{L^2 T^2}\right) \leqslant C(p)\left(\frac{(1+\frac{1}{\delta})^4}{R^4} + \frac{1}{(M^{p-1}\delta)^2 T^2}\right),$$

which implies

$$\frac{|\nabla g|^2}{(\alpha - g)^2} \leqslant C(p)\left(\frac{(1+\frac{1}{\delta})^2}{R^2} + \frac{1}{M^{p-1}\delta T}\right).$$

Then the conclusion of Theorem 3.3 follows easily from the fact that

$$\frac{|\nabla g|}{\alpha - g} = \frac{|\nabla G(u)|}{\alpha - G(u)}. \qquad \square$$

An immediate application of this theorem is the following time-dependent Liouville theorem for PME on **R**.

Theorem 3.4 *(Liouville theorem). Let u be a positive ancient solution, a solution defined in all space and negative time, to the Porous Medium Equation ($p > 1$) on* **R***, such that*

$$u(x,t) = o\big(d(x)^{1/(p-1)} + |t|^{1/(p-1)}\big)$$

near infinity. Then u is a constant.

Proof. By our assumption, the function u satisfies $u(x,t) = o(d(x)^{1/(p-1)} + |t|^{1/(p-1)})$ near infinity. Fixing (x_0, t_0) in space–time and using Theorem 3.3 for u on the cube $Q(R,R) = B(x_0, R) \times [t_0 - R, t_0]$, we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)} \leqslant C(p)\left(\frac{1+\delta}{\delta R} + \frac{1}{\sqrt{M^{p-1}\delta T}}\right)M^{p-1} \leqslant C(p,\delta)\left(\frac{M^{p-1}}{R} + \sqrt{\frac{M^{p-1}}{T}}\right),$$

where M is the maximum value on the double cube $Q(2R, 2R)$, by our assumption on the growth condition of the function u at infinity,

$$M_{2R,2R} = o\big(R^{1/(p-1)} + R^{1/(p-1)}\big) = o\big(R^{1/(p-1)}\big). \tag{29}$$

Hence we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)} \leqslant C(p,\delta)\left(\frac{M^{p-1}}{R} + \sqrt{\frac{M^{p-1}}{R}}\right) = o(1). \tag{30}$$

Letting $R \to \infty$, it follows that $|\nabla u(x_0,t_0)| = 0$. Since (x_0,t_0) is arbitrary, one sees that u must be a constant function. □

For dimension $n \geqslant 2$, we have the following localized Hamilton-type gradient estimates for the positive solution of PME on $M^n \times (-\infty, \infty)$:

Theorem 3.5. *Let M^n be a complete Riemannian manifold with dimension $n \geqslant 2$, $Ric(M^n) \geqslant -k$, $k \geqslant 0$. Suppose that u is a positive solution of the PME*

$$u_t = \Delta\big(u^p\big)$$

in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T, t_0] \subset M^n \times (-\infty,\infty)$, with range$(u) = [m, M]$. *Let $G(u) = \frac{p}{p-1}u^{p-1}$, $\alpha = \frac{p}{p-1}M^{p-1}(1+\delta)$ with some small constant $0 < \delta \leqslant \frac{4}{n-1}$. If the following* pinch condition *on m, M holds*

$$1 \leqslant \left(\frac{M}{m}\right)^{p-1} < \frac{1}{1+\delta}\left(\frac{4p}{(n-1)(p-1)} + 1\right), \tag{31}$$

then there exists a constant $C(n,p)$ depending only on n and p, and

$$\gamma = 2p - \frac{(n-1)(p-1)}{2} \times \frac{M^{p-1}(1+\delta) - m^{p-1}}{m^{p-1}} > 0,$$

such that

$$\frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leqslant C(n,p)\left(\frac{\delta+1}{\gamma\delta R} + \frac{1}{\sqrt{\gamma\delta M^{p-1}T}} + \sqrt{\frac{k}{\delta}}\right)$$

for all (x,t) in $Q_{R/2,T/2}$.

Proof. We have $F(s) = s^p$ and choose $G(s) = \frac{p}{p-1}s^{p-1}$ in Theorem 1.1. We have $F'(s) = ps^{p-1}$, $F''(s) = p(p-1)s^{p-2}$, and $K = pM^{p-1}$, and α, as defined above, then conditions (A) and (B) are satisfied and condition (C) in Theorem 1.1 becomes

$$\begin{cases} \tau \geqslant p-1, \\ 2p - \dfrac{(n-1)(p-1)}{2} \times \dfrac{M^{p-1}(1+\delta) - s^{p-1}}{s^{p-1}} \geqslant \gamma > 0, \quad \forall s \in [m, M], \end{cases}$$

which is equivalent to

$$\begin{cases} \tau \geqslant p-1, \\ 2p - \dfrac{(n-1)(p-1)}{2} \times \dfrac{M^{p-1}(1+\delta) - m^{p-1}}{m^{p-1}} > 0. \end{cases}$$

Let $\tau = p-1$ and $\gamma = 2p - \frac{(n-1)(p-1)}{2}\frac{M^{p-1}(1+\delta)-m^{p-1}}{m^{p-1}} > 0$; the above condition as $0 < \delta \leqslant \frac{4}{n-1}$ is equivalent to our pinch condition (31).

Similar to the proof of Theorem 3.3, we have

$$\begin{cases} g = \dfrac{p}{p-1}u^{p-1}, \quad g' = pu^{p-1}, \quad g'' = p(p-1)u^{p-1}, \\ f = \dfrac{2g'}{\alpha - g} = 2(p-1)\dfrac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leqslant \dfrac{2(p-1)}{\delta}, \\ b = \dfrac{g''}{g'} = p-1, \\ L = \gamma(\alpha - g) = \dfrac{p\gamma}{1-p}\big[M^{p-1}(1+\delta) - u^{p-1}\big] \geqslant \dfrac{p\gamma}{1-p}M^{p-1}\delta, \\ L_1 = 1 + p + f \leqslant 1 + p + 2(p-1)\dfrac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leqslant 1 + p + \dfrac{2(p-1)}{\delta}. \end{cases}$$

By the same argument in the proof of Theorem 3.3, we have

$$\Psi w^2 \leqslant C(n,p)\left(\frac{(\delta+1)^4}{\gamma^4\delta^4R^4} + \frac{1}{\gamma^2\delta^2M^{2(p-1)}T^2} + \frac{k^2}{\delta^2}\right)$$

which implies

$$\frac{|\nabla g|^2}{(\alpha - g)^2} \leqslant C(n,p)\left(\frac{(\delta+1)^2}{\gamma^2\delta^2R^2} + \frac{1}{\gamma\delta M^{p-1}T} + \frac{k}{\delta}\right).$$

Then the conclusion of Theorem 3.5 follows easily from the fact that

$$\frac{|\nabla g|}{\alpha - g} = \frac{|\nabla G(u)|}{\alpha - G(u)}. \qquad \square$$

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Appendix A

Here we show a technical linear algebra lemma for symmetric matrix.

Lemma A.1. *Let $A = (a_{ij})$ be a nonzero $n \times n$ symmetric matrix with eigenvalues $\{\lambda_k\}$, for any $a, b \in \mathbf{R}$, one has the following properties*:

$$\text{(a)} \quad |A|^2 = \sum_{i,j=1}^{n} a_{ij}^2 = \operatorname{tr}\bigl(AA^T\bigr) = \sum_{k=1}^{n} \lambda_k^2,$$

$$\text{(b)} \quad \max_{|v|=1} (aA + b \operatorname{tr} A I_n)(v, v) = a\lambda_i + b \sum_{k=1}^{n} \lambda_k, \quad \textit{for some } 1 \leqslant i \leqslant n,$$

$$\min_{|v|=1} (aA + b \operatorname{tr} A I_n)(v, v) = a\lambda_j + b \sum_{k=1}^{n} \lambda_k, \quad \textit{for some } 1 \leqslant j \leqslant n,$$

$$\text{(c)} \quad \max_{A \in S(n);\, |v|=1} \left[\frac{aA + b \operatorname{tr} A I_n}{|A|}(v, v) \right]^2 = (a+b)^2 + (n-1)b^2.$$

Proof. (a) follows from direct computation and A symmetry.

(b) follows from the facts that

$$\max_{|v|=1} (aA + b \operatorname{tr} A I_n)(v, v), \qquad \min_{|v|=1} (aA + b \operatorname{tr} A I_n)(v, v)$$

are the maximal and minimal eigenvalue of $aA + b \operatorname{tr} A I_n$, and the eigenvalues of $aA + b \operatorname{tr} A I_n$ are $\{a\lambda_i + b \sum_{k=1}^{n} \lambda_k\}_{i=1}^{n}$.

To prove (c), applying (a) and (b), we have

$$\max_{A \in S(n);\, |v|=1} \left[\frac{aA + b \operatorname{tr} A I_n}{|A|}(v, v) \right]^2 = \max_{\sum_{k=1}^{n} \lambda_k^2 = 1} \left[a\lambda_1 + b \sum_{k=1}^{n} \lambda_k \right]^2.$$

By Lagrangian multiplier method in calculus, the extremums of

$$f(\lambda_1, \ldots, \lambda_n) = a\lambda_1 + b \sum_{k=1}^{n} \lambda_k$$

under constrain $\sum_{k=1}^{n} \lambda_k^2 = 1$ are

$$-\bigl[(a+b)^2 + (n-1)b^2\bigr]^{1/2} \leqslant a\lambda_1 + b \sum_{k=1}^{n} \lambda_k \leqslant \bigl[(a+b)^2 + (n-1)b^2\bigr]^{1/2}. \qquad \square$$

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